Simultaneous Development of Velocity and Temperature Distributions in a Flat Duct with Uniform Wall Heating

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Laminar forced-convection heat transfer in a parallel-plate channel (flat duct) with uniform heat flux at the walls is analyzed. The velocity and temperature distributions, both uniform at the entrance section, develop simultaneously as the fluid flows through the duct. The heat transfer results, obtained for the Prandtl-number range of 0.01 to 50, include the Nusselt-number variation along the channel and the wall-temperature variation corresponding to the prescribed uniform heat flux.

Forced-convection heat transfer in the thermal entrance region of tubes and ducts has been a subject of analytical study since the pioneering work of Graetz (1883). In most instances consideration has been given to the entrance of the fluid into the heated (or cooled) region of the duct with an already fully developed velocity profile which is unchanging along the duct length. This simplification is not realistic when the fluid enters the heated channel directly from a reservoir or large header. In such a situation the velocity is more nearly uniform across the entrance section, and the velocity and thermal boundary layers develop simultaneously as the fluid moves through the tube.

For laminar flow, which is the condition considered in this paper, the problem of simultaneous development of velocity and thermal boundary layers has been solved for a circular tube and flat duct under the condition of uniform wall temperature (*1* to *4*). For uniform wall heat flux results are available only for the circular tube (*1*). The purpose of the present study is to analyze the uniform wall heat-flux situation in a parallel-plate channel (often referred to as a *flat duct*). The heat transfer results to be presented include the Nusselt-number variation along the channel and the wall-temperature variation corresponding to the prescribed uniform heat flux.

ANALYSIS

Physical Model

In Figure 1 fluid is pictured as flowing from left to right through a parallel-plate channel of half width a. Both the temperature and velocity are assumed to be uniform across the entrance section ($x = 0$). A uniform heat flux is maintained at the channel walls. As the fluid progresses down the channel, the respective action of viscosity and heat conduction at the wall causes vorticity and heat to spread from the channel walls into the main stream. The distance (away from the wall) to which these effects propagate into the main stream can be measured in terms of boundary-layer thicknesses. Depending upon the fluid properties, the velocity and thermal effects diffuse at different rates, and hence the thicknesses of the velocity and thermal boundary layers will generally be different. Figure 2 illustrates schematically how the boundary layers develop in the entrance region of a duct. Figure 2*a* refers to low-Prandtl-number fluids where $\Delta > \delta$, Figure 2*b* is for high Prandtl numbers where $\Delta < \delta$.

The temperature of the fluid outside the thermal boundary layer is essentially the same as the entering temperature; however because the fluid near the channel walls is retarded owing to viscosity, the velocity outside the velocity boundary layer must increase along the channel length to preserve the same total mass flow. Within the boundary layers the temperature and velocity vary with both x and y.

Far down the channel, when both the velocity and thermal boundary layers have completely filled the section, it is well established that the local heat transfer coefficient becomes a constant. For this fully developed thermal condition the temperature distribution across the channel can be calculated to be

$$\frac{T - T_{CL}}{\frac{qa}{k}} = \frac{5}{8}\left[1 - \frac{8}{5}\left(\frac{y}{a}\right) + \frac{4}{5}\left(\frac{y}{a}\right)^3 - \frac{1}{5}\left(\frac{y}{a}\right)^4\right] \quad (1a)$$

and the fully developed heat transfer coefficient based on the definition $q = h(T_w - T_b)$ is given by

$$\frac{ha}{k} = \frac{35}{17} \quad (1b)$$

Mathematical Solution

To compute entrance-region heat transfer coefficients one begins with the equation expressing conservation of energy within the fluid. A control surface situated in the thermal boundary layer (Figure 3) will be considered. Eckert (*5*) has shown that conservation of energy for this element is

$$\rho c_p \frac{d}{dx}\left[\int_0^\Delta u(T - T_e)\, dy\right] = q \quad (2)$$

The left side of Equation (2) represents the convective transport of heat; the right side is the wall heat transfer, which is prescribed to be uniform along the channel for the present problem.

Equation (2) represents a means of solving for the temperature distribution $T(x, y)$, provided that the velocity distribution $u(x, y)$ is known. Fortunately information on the development of the velocity boundary layer in a parallel-plate channel is already available. An approximate solution was first formulated by Schiller (*4*), who gives the velocity profile as

$$u = U_s\left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right], \quad 0 \le y \le \delta \quad (3a)$$

$$u = U_s \quad \delta \le y \le a \quad (3b)$$

where both U_s and δ vary with x. Once δ and U_s are known, the velocity distribution is completely determined. These functions have been computed (*4*) and are used in the present investigation.

With the velocity distribution specified, Equation (2) can be used to solve for the temperature distribution in the boundary layer. This solution is readily found by expressing the temperature as a polynomial in y the coefficients of which are unknown functions of x. Then the unknown coefficients are found from Equation (2). A form for the polynomial which immediately suggests itself is that in Equation (1*a*). This equation is modified to the following form, so that the fluid takes on the temperature T_e at the edge of the thermal boundary layer

$$T - T_e = \frac{5}{8}\frac{q\Delta}{k}\left[1 - \frac{8}{5}\left(\frac{y}{\Delta}\right) + \frac{4}{5}\left(\frac{y}{\Delta}\right)^3 - \frac{1}{5}\left(\frac{y}{\Delta}\right)^4\right], \quad 0 \le y \le \Delta \quad (4a)$$

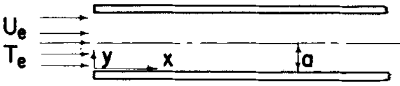

Fig. 1. Geometric configuration and coordinate system.

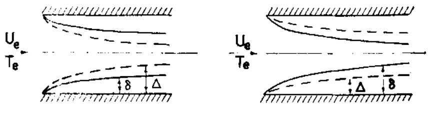

Fig. 2a. Boundary layer growth for low Prandtl numbers.
Fig. 2b. Boundary layer growth for high Prandtl numbers.

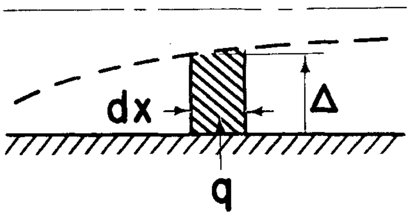

Fig. 3. Control volume for deriving energy equation.

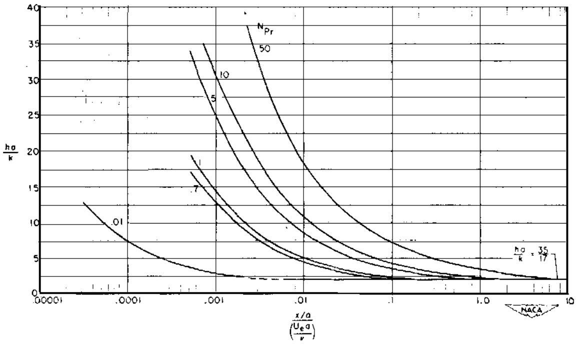

Fig. 4. Nusselt number variation along the duct length.

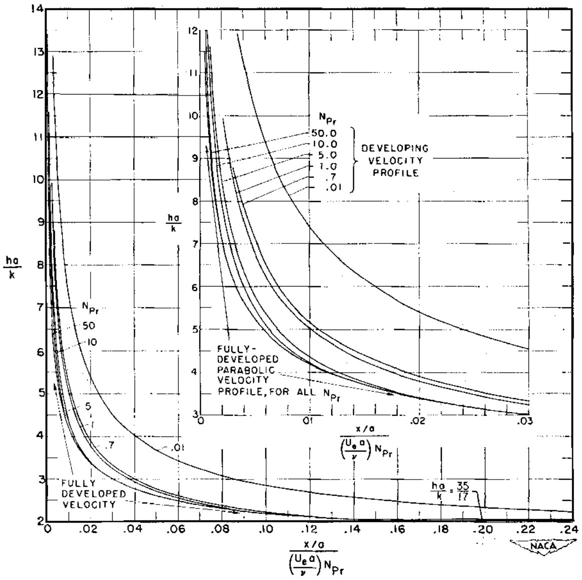

Fig. 5. Effect of developing velocity profile on Nusselt number variation.

$$T = T_e \qquad \Delta \leq y \leq a \qquad (4b)$$

The only unknown appearing in Equation (4a) is the thermal-boundary-layer thickness; once this function has been determined [with the aid of Equation (2)], the temperature distribution is completely specified for every x and y. It is worth noting that when the thermal boundary layer fills the channel, that is $\Delta = a$, Equation (4a) coincides with the fully developed temperature distribution (1a)

To find the variation of Δ with x, Equations (3a), (3b), (4a), and (4b) are introduced into the energy equation (2). After the indicated operations have been carried out, a pair of algebraic equations for computing Δ as a function of x results:

$$\frac{\bar{\Delta}^3}{\bar{\delta}}\left[\frac{14}{75} - \frac{4}{105}\frac{\bar{\Delta}}{\bar{\delta}}\right] = \frac{8}{5N_{Pr}}\frac{\bar{x}}{\bar{U}_s}, \qquad \Delta < \delta \qquad (5a)$$

$$\bar{\Delta}^2\left[\frac{9}{25} - \frac{1}{3}\frac{\bar{\delta}}{\bar{\Delta}} + \frac{2}{15}\left(\frac{\bar{\delta}}{\bar{\Delta}}\right)^2 - \frac{1}{75}\left(\frac{\bar{\delta}}{\bar{\Delta}}\right)^4 + \frac{1}{525}\left(\frac{\bar{\delta}}{\bar{\Delta}}\right)^5\right] = \frac{8}{5N_{Pr}}\frac{\bar{x}}{\bar{U}_s}, \qquad \Delta > \delta \qquad (5b)$$

Since $\bar{\delta}$ and $\bar{U}_s$ are known functions of $\bar{x}$, the variation of $\bar{\Delta}$ with $\bar{x}$ can be calculated from these equations for a fluid with any Prandtl number. For high Prandtl numbers Equation (5a) applies, and for low Prandtl numbers Equation (5b) is used. The values of $\bar{\Delta}$ are then used to evaluate the heat transfer results.

RESULTS

Heat Transfer Coefficients

The heat transfer results will be presented in terms of a local heat transfer coefficient defined as

$$h = \frac{q}{T_w - T_b} \qquad (6)$$

The local wall temperature may be evaluated from Equation (4a) by setting $y = 0$, with the result

$$T_w - T_e = \frac{5}{8}\frac{q\Delta}{k} \qquad (7)$$

where the thermal boundary layer thickness has been computed as described in the previous section. The local bulk temperature can be written directly from an over-all energy balance by equating the enthalpy increase of the fluid to the total heat transfer at the wall. This gives

$$T_b - T_e = \frac{q}{\rho a c_p U_e} x \qquad (8)$$

The temperature difference in Equation (6) can then be evaluated in terms of Equations (7) and (8); the following is the result for the Nusselt number:

$$N_{Nu} = \frac{ha}{k} = \frac{1}{\frac{5}{8}\left(\frac{\Delta}{a}\right) - \frac{x/a}{\left(\frac{U_e a}{\nu}\right)N_{Pr}}} \qquad (9)$$

By means of the calculated values of Δ/a (which depend on $(x/a)/(U_e a/\nu)$ and N_{Pr}), this dimensionless local heat transfer result has been computed and plotted on Figure 4. For a given flow situation it is seen that with increasing x the Nusselt numbers approach the fully developed value of 35/17. For fluids with low Prandtl numbers the fully developed heat transfer condition is approached much more rapidly than for fluids with high Prandtl numbers. This is a consequence of the fact that the thermal boundary layer develops more rapidly for low-Prandtl-number fluids. Similar results have been obtained for laminar flow in tubes and ducts having uniform wall temperature.

The influence of the developing velocity profile on heat transfer will now be examined. To accomplish this, a comparison is made with the analysis carried out for a velocity distribution which is fully developed at the duct entrance and unchanging along the duct length (Figure 5). Figure 5 is constructed in two parts to permit better examination of the entire range of the abscissa. The heat transfer results, corresponding to the fully developed velocity profile, plot as a single curve for all Prandtl numbers when presented with this abscissa variable. It is seen that the lower the Prandtl number the greater the effect of the developing velocity distribution. In all

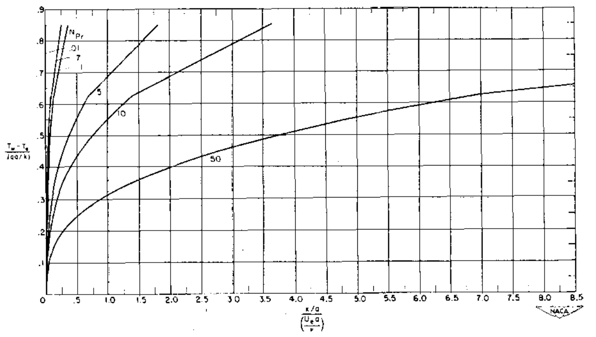

Fig. 6. Wall temperature variation along the duct length.

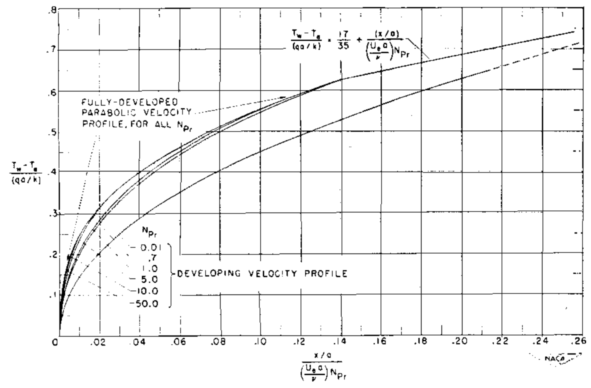

Fig. 7. Effect of developing velocity profile on wall temperature variation.

cases the simultaneous velocity development causes an increase in heat transfer relative to the situation of unchanging velocity distribution.

From Figures 4 and 5 one can obtain an estimate of the length of duct required to establish fully developed heat transfer conditions. This gives

$$(x/a)_{ent} = 0.14 \left(\frac{U_e a}{\nu}\right) N_{Pr} \qquad N_{Pr} \geq 0.7 \qquad (10a)$$

$$(x/a)_{ent} = 0.003 \frac{U_e a}{\nu} \qquad N_{Pr} = 0.01 \qquad (10b)$$

The approximate nature of the mathematical formulation does not permit precise calculations for thermal entrance length.

Wall-Temperature Variation

An alternate presentation of the results may be made in terms of the wall-temperature variation along the channel (Figures 6 and 7). Since the wall heat transfer rate is a prescribed constant in the present problem, the important unknown in a practical situation would be the wall temperature. On each curve there is a change in slope at the end of the thermal entrance region, because of the simplification in the analysis. In reality the transition from the entrance region to the fully developed heat transfer condition would be smooth. In the fully developed region the wall-temperature variation is given by

$$\frac{T_w - T_e}{qa/k} = \frac{17}{35} + \frac{x/a}{\left(\frac{U_e a}{\nu}\right) N_{Pr}} \qquad (11)$$

This relation can be used when x/a exceeds the entrance lengths given by Equations (10*a*) and (10*b*).

On Figure 7 the wall-temperature variations are compared with the corresponding results for the case of a fully developed, unchanging velocity profile. It is seen that the simultaneous development of the velocity profile tends to lower the wall temperature in the entry region, with the effect increasing with decreasing Prandtl number. On the abscissa scale of Figure 7 the results for $N_{Pr} = 0.01$ may be clearly read, and on Figure 6 the curve for this Prandtl number is crowded against the ordinate axis and hence does not appear.

CONCLUDING REMARK

While the analysis presented here is for a parallel-plate channel, the results are expected to apply also for annuli where the radius ratio is close to unity.

ACKNOWLEDGMENT

The authors would like to acknowledge the helpful comments by Thomas F. Irvine, Jr., of the University of Minnesota, who originally suggested this problem.

NOTATION

a = half width of parallel-plate channel
c_p = specific heat at constant pressure
h = local coefficient of heat transfer, $q/(T_w - T_b)$
k = thermal conductivity of fluid
N_{Nu} = local Nusselt number, ha/k
N_{Pr} = Prandtl number, $c_p\mu/k = \nu/\alpha$
q = local heat flux per unit area at channel walls
T = temperature; T_b, bulk fluid temperature; T_e, uniform temperature of entering fluid (equal to fluid temperature external to thermal boundary layer); T_w, wall temperature; T_{CL}, temperature at channel center line
U_e = uniform velocity of entering fluid
U_s = fluid velocity external to velocity boundary layer
$\bar{U}_s$ = dimensionless velocity, U_s/U_e
u = fluid velocity
x = axial coordinate measured from channel entrance
$\bar{x}$ = dimensionless axial coordinate, $(x/a)/(U_e a/\nu)$
y = transverse coordinate measured from channel wall

Greek Letters

α = thermal diffusivity, $k/\rho c_p$
Δ = thermal-boundary-layer thickness
$\bar{\Delta}$ = dimensionless thermal-boundary-layer thickness, Δ/a
δ = velocity-boundary-layer thickness
$\bar{\delta}$ = dimensionless velocity-boundary-layer thickness, δ/a
μ = absolute viscosity
ν = kinematic viscosity
ρ = fluid density

Subscript

ent = thermal entrance length

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Manuscript received May 19, 1958; revision received July 25, 1958; paper accepted July 28, 1958.